Axiomatising Divergence*

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Abstract. This paper develops sound and complete axiomatisations for the divergence sensitive spectrum of weak bisimulation equivalence. The axiomatisations can be extended to a considerable fragment of the linear time – branching time spectrum with silent moves, partially solving an open problem posed in [5].

1 Motivation

The study of comparative concurrency semantics is concerned with a uniform classification of process behaviour, and has cumulated in Rob van Glabbeek's seminal papers on the *linear time-branching time spectrum* [4,5]. The main ('vertical') dimension of the spectrum with silent moves [5] spans between trace equivalence (TE) and branching bisimulation (BB), and identifies different ways to discriminate processes according to their branching structure, where BB induces the finest, and TE the coarsest reasonable semantics. Due to the presence of silent moves, this spectrum is spread in another ('horizontal') dimension, determined by the semantics of *divergence*. In the fragment spanning from weak bisimulation (WB) to BB, seven different criteria to distinguish divergence induce a 'horizontal' lattice, and this lattice appears for all the bisimulation relations.

To illustrate the spectrum, van Glabbeek lists a number of examples and counterexamples showing the differences among the various semantics [5]. *Process algebra* provides a different – and to our opinion more elegant – way to compare semantic issues, by providing distinguishing axioms that capture the essence of an equivalence (or preorder). For the 'vertical' dimension of the spectrum, these distinguishing axioms are well-known (see e.g. [4,7,2]). However, the 'horizontal' dimension has resisted an axiomatic treatment so far. We believe that this is mainly due to the fact that divergence only makes sense in the presence of recursion, and that recursion is hard to tackle axiomatically. Isolated points in the 'horizontal' dimension have however been axiomatised, most

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notably Milner's weak bisimulation (WB) congruence [10], and also convergent WB preorder [11], as well as divergence insensitive BB congruence [6] and stable WB congruence [8]. It is also worth to mention the works of [3] and [1], which axiomatised divergence sensitive WB congruence and convergent WB preorder, respectively, but without showing completeness in the presence of recursion.

This paper develops complete axiomatisations for the 'horizontal' dimension of weak bisimulation equivalence. A lattice of distinguishing axioms is shown to characterise the distinct semantics of divergence, and to precisely reflect the 'horizontal' lattice structure of the spectrum. We are confident that these axioms form the basis of complete axiomatisation for the bisimulation spectrum spanning from WB to BB.

The paper is organised as follows. Section 2 introduces the necessary notation and definitions, while Section 3 recalls the weak bisimulation equivalences and Section 4 introduces the axiom systems. Section 5 is devoted to soundness of the axioms and sets the ground for the completeness proof. Section 6 is devoted to the main step of the proof, only focusing on closed expressions, while Section 7 covers open expressions. Section 8 concludes the paper. Proofs that are omitted in this extended abstract will appear in the extended version of this paper.

2 Preliminaries

We assume a set of variables $\mathbb{V}$, and a set of actions $\mathbb{A}$, containing the silent action τ. We consider the set of open finite state agents with silent moves and explicit divergence, given as the set $\mathbb{E}$ of expressions generated by the grammar

$$\mathcal{E} \quad ::= \quad a.\mathcal{E} \quad | \quad \mathcal{E}+\mathcal{E} \quad | \quad recX.\mathcal{E} \quad | \quad X \quad | \quad \Delta(\mathcal{E})$$

where $X \in \mathbb{V}$ and $a \in \mathbb{A}$. $\Delta(E)$ is an expression that adds divergence explicitly to the root of E. It can be considered as a syntactic shorthand for $recX.(\tau.X+E)$ provided X does not occur in E. The explicit representation of divergence by means of Δ will prove handy in the sequel.

The syntactic equality on $\mathbb{E}$ is denoted by $\equiv$. With $\mathbb{V}(E)$ we denote the set of all variables that are free in $E \in \mathbb{E}$, i.e., not bounded by a $recX$-operator. We define $\mathbb{P} = \{E \in \mathbb{E} \mid \mathbb{V}(E) = \emptyset\}$. We use $E, F, G, H, \ldots$ (resp. $P, Q, R, \ldots$) to range over expressions from $\mathbb{E}$ (resp. $\mathbb{P}$). If $\boldsymbol{F} = F_1, \ldots, F_n$ is a sequence of expressions, $\boldsymbol{X} = X_1, \ldots, X_n$ is a sequence of variables, and $E \in \mathbb{E}$ then $E\{\boldsymbol{F}/\boldsymbol{X}\}$ denotes the expression that results from E by simultaneously replacing all free occurrences of X_i in E by F_i $(1 \leq i \leq n)$. The variable X is *guarded* in E, if every free occurrence of X in E lies within a subexpression of the form $a.F$ with $a \in \mathbb{A}\backslash\{\tau\}$, otherwise X is called unguarded in E. E is guarded if for every subexpression $recY.F$ of E the variable Y is guarded in F.

The semantics of $\mathbb{E}$ is given as the least transition relation satisfying the following rules, which are standard (except that, as indicated before, $\Delta(E)$ can diverge, in addition to exhibiting all the behaviour of E).

$$\frac{}{a.E \xrightarrow{a} E} \qquad \frac{E \xrightarrow{a} E'}{E+F \xrightarrow{a} E'} \qquad \frac{E \xrightarrow{a} E'}{F+E \xrightarrow{a} E'}$$

$$\frac{E\{recX.E/X\} \xrightarrow{a} E'}{recX.E \xrightarrow{a} E'} \qquad \frac{E \xrightarrow{a} E'}{\Delta(E) \xrightarrow{a} E'} \qquad \frac{}{\Delta(E) \xrightarrow{\tau} \Delta(E)}$$

3 The Bisimulations

Since we are working in the context of silent steps, we define a few standard abbreviations: $E \Longrightarrow F$ if $E \xrightarrow{\tau}{}^{*} F$; $E \stackrel{a}{\Longrightarrow} F$ if $E \Longrightarrow \xrightarrow{a} \Longrightarrow F$; $E \stackrel{\widehat{a}}{\Longrightarrow} F$ if ($E \stackrel{a}{\Longrightarrow} F$ and $a \neq \tau$) or ($E \Longrightarrow F$ and $a = \tau$). We write $E \xrightarrow{a}$ (resp. $E \longrightarrow$) if $E \xrightarrow{a} F$ for some $F \in \mathbb{E}$ (resp. $E \xrightarrow{a} F$ for some $a \in \mathbb{A}$, $F \in \mathbb{E}$). With $E \stackrel{a}{\not\longrightarrow}$ and $E \not\longrightarrow$ we denote the corresponding negated conditions. We let $E \Uparrow$ denote $E \xrightarrow{\tau}{}^{\omega}$, i.e., E has the possibility to diverge. Finally, $E \Lsh$ denotes that there is some F such that $E \Longrightarrow F$ and either $F \Uparrow$ or $F \not\longrightarrow$ (or equivalently, $E \Uparrow$ or $E \Longrightarrow F \not\longrightarrow$ for some F), i.e., E may either diverge, or silently decide to terminate. For a relation $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ define the following conditions (in all conditions $P, Q, P' \in \mathbb{P}$ and $a \in \mathbb{A}$ are implicitly $\forall$-quantified):

(WB) if $(P,Q) \in \mathcal{R} \wedge P \xrightarrow{a} P'$ then $Q \stackrel{\widehat{a}}{\Longrightarrow} Q' \wedge (P',Q') \in \mathcal{R}$ for some Q',
(S) if $(P,Q) \in \mathcal{R} \wedge P \stackrel{\tau}{\not\longrightarrow}$ then $Q \Longrightarrow Q' \stackrel{\tau}{\not\longrightarrow}$ for some Q',
(0) if $(P,Q) \in \mathcal{R} \wedge P \not\longrightarrow$ then $Q \Longrightarrow Q' \not\longrightarrow$ for some Q',
(Δ) if $(P,Q) \in \mathcal{R} \wedge P \Uparrow$ then $Q \Uparrow$,
(λ) if $(P,Q) \in \mathcal{R} \wedge P \Lsh$ then $Q \Lsh$.

Let $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ be a symmetric relation. We say that $\mathcal{R}$ is a

- *weak bisimulation* (WB^{ϵ} or simply WB) if $\mathcal{R}$ satisfies (WB).
- *stable weak bisimulation* (WB^{S}) if $\mathcal{R}$ satisfies (WB) and (S).
- *completed weak bisimulation* (WB^{0}) if $\mathcal{R}$ satisfies (WB) and (0).
- *divergent weak bisimulation* (WB^{λ}) if $\mathcal{R}$ satisfies (WB) and (λ).
- *divergent stable weak bisimulation* (WB^{Δ}) if $\mathcal{R}$ satisfies (WB) and (Δ).

In the sequel, we let $*$ range over the set $\{\Delta, \lambda, S, 0, \epsilon\}$. The relation $\sim^{*} \subseteq \mathbb{P} \times \mathbb{P}$ is defined as the union of all WB^{*}, it is easily seen to be itself a WB^{*} as well as an equivalence relation.

Theorem 1. [5] *The equivalences $\sim^{*}$ are ordered by inclusion according to the lattice in Figure 1. The upper relation contains the lower if and only if both are connected by a line.*

Examples that distinguish these equivalences can be found in [5]. It is a well known deficiency that $\sim^{*}$ is not a congruence w.r.t. '+', moreover, for $* \in \{\Delta, \lambda, S, 0\}$ it is not a congruence w.r.t. $\Delta(.)$. For instance $\tau.0 \sim^{\Delta} 0$, but $\Delta(\tau.0) \not\sim^{\Delta} \Delta(0)$. To obtain the coarsest congruences in $\sim^{*}$ on $\mathbb{P}$, we define each $\simeq^{*}$ to be the relation that contains exactly the pairs $(P,Q) \in \mathbb{P} \times \mathbb{P}$ that satisfy the following *root conditions*:

- if $P \xrightarrow{a} P'$ then $Q \stackrel{a}{\Longrightarrow} Q'$ and $P' \sim^{*} Q'$ for some Q'
- if $Q \xrightarrow{a} Q'$ then $P \stackrel{a}{\Longrightarrow} P'$ and $P' \sim^{*} Q'$ for some P'

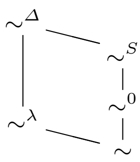

Fig. 1. Inclusions between the relations $\sim^*$

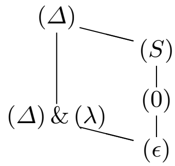

Fig. 2. Implications between the distinguishing axioms

We lift these relations from $\mathbb{P}$ to $\mathbb{E}$ as usual: let $E, F \in \mathbb{E}$, and let $\boldsymbol{X} = X_1, \ldots, X_n$ be a sequence of variables that contains all variables in $\mathbb{V}(E) \cup \mathbb{V}(F)$. Then $E \simeq^* F$ if $E\{\boldsymbol{P}/\boldsymbol{X}\} \simeq^* F\{\boldsymbol{P}/\boldsymbol{X}\}$ for all $\boldsymbol{P} = P_1, \ldots, P_n$ with $P_i \in \mathbb{P}$ (analogously for $\sim^*$).

Theorem 2. *The relation $\simeq^*$ is the coarsest congruence contained in $\sim^*$ w.r.t. the operators of $\mathbb{E}$. All inclusions from Figure 1 carry over from $\sim^*$ to $\simeq^*$.*

4 Axioms

This section introduces a lattice of axioms characterising the above weak bisimulations. For $* \in \{\Delta, S, 0, \epsilon\}$, the axioms for $\simeq^*$ are given in Table 1, plus the axiom $(*)$ from Table 2. The axioms for $\simeq^\lambda$ are given in Table 1, plus the axioms (Δ) and (λ) from Table 2. We write $E =^* F$ if $E = F$ can be derived by application of the axioms for $\simeq^*$.

The axioms from Table 1 are standard [10] except of (*rec*5) and (*rec*6). Axiom (*rec*5) makes divergence explicit if introduced due to silent recursion; it defines the nature of the Δ-operator. Axiom (*rec*6) states the redundancy of recursion on an unguarded variable in the context of divergence.

We discuss the distinguishing axioms in reverse order relative to how they are listed in Table 2. Axiom (λ) characterises the property of WB^λ that divergence cannot be distinguished when terminating. Axiom (ϵ) represents Milner's 'fair' setting, where divergence is never distinguished. The remaining three axioms state that divergence cannot be distinguished if the process can still perform an action to escape the divergence (0), that it cannot be distinguished if the process can perform a silent step to escape divergence (S), and that two consecutive divergences cannot be properly distinguished (Δ). It is a simple exercise to verify the implications between the distinguishing axioms as summarized in the lattice in Figure 2. It nicely reflects the inclusions between the respective congruences. The upper axioms turn into derivable laws given the lower ones (plus the core axioms from Table 1) as axioms.

The following two Δ-unfolding laws can be derived from the axioms for $\simeq^\Delta$ (and thus for all $\simeq^*$), they will be useful in Section 6.

$$(\tau\Delta)\ \ \Delta(E) =^\Delta \tau.\Delta(E) + E \qquad\qquad (\tau\Delta')\ \ \Delta(E) =^\Delta \tau.\Delta(E)$$

5 Soundness and Completeness

Checking soundness of the axioms is tedious but follows standard techniques.

Table 1. Core axioms

$(S1)\ E + F = F + E$	$(\tau 1)\ a.\tau.E = a.E$
$(S2)\ E + (F + G) = (E + F) + G$	$(\tau 2)\ \tau.E + E = \tau.E$
$(S3)\ E + E = E$	$(\tau 3)\ a.(E + \tau.F) = a.(E + \tau.F) + a.F$
$(S4)\ E + 0 = E$	

$(rec1)$ if Y is not free in $recX.E$ then $recX.E = recY.(E\{Y/X\})$
$(rec2)$ $recX.E = E\{recX.E/X\}$
$(rec3)$ if X is guarded in E and $F = E\{F/X\}$ then $F = recX.E$
$(rec4)$ $recX.(X + E) = recX.E$
$(rec5)$ $recX.(\tau.(X + E) + F) = recX.\Delta(E + F)$
$(rec6)$ $recX.(\Delta(X + E) + F) = recX.\Delta(E + F)$

Table 2. Distinguishing axioms

(Δ)	$\Delta(\Delta(E) + F) = \tau.(\Delta(E) + F)$
(S)	$\Delta(\tau.E + F) = \tau.(\tau.E + F)$
(0)	$\Delta(a.E + F) = \tau.(a.E + F)$
(ϵ)	$\Delta(E) = \tau.E$
(λ)	$\Delta(0) = \tau.0$

Theorem 3 (soundness). *If $E, F \in \mathbb{E}$ and $E =^* F$ then $E \simeq^* F$.*

In order to show completeness, i.e., that $E \simeq^* F$ implies $E =^* F$, we proceed along the lines of [10], except for the treatment of expressions from $\mathbb{E} \setminus \mathbb{P}$. We will work as much as possible in the setting of $W\!B^{\Delta}$, the finest setting. As in [10] the first step consists in transforming every expression into a guarded one:

Theorem 4. *Let $E \in \mathbb{E}$. There exists a guarded F with $E =^{\Delta} F$.*

We do not consider $* = \epsilon$ in the sequel because by using axiom (ϵ), for every $E \in \mathbb{E}$ we find an E' such that E' does not contain the Δ-operator and $E =^{\epsilon} E'$. This allows to apply Milner's result [10] that in the absence of the Δ-operator the axioms from Table 1 with $(rec5)$ and $(rec6)$ replaced by Milner's rec-laws ($recX(\tau.X + E) = recX(\tau.E)$ and $recX.(\tau.(X + E) + F) = recX.(\tau.X + E + F)$, both can be easily derived from $(rec5)$ and (ϵ)) are complete for $\simeq^{\epsilon}$.

The basic ingredients of the completeness proof are equation systems, and the manner in which these systems are set up constitutes the crucial deviation from the proof of Milner. Before we give detailed account of the proof, we illustrate the strategy by a small, informal example.

Consider an equation such as $X = a.X$. This equation is said to have a unique solution modulo Milner's observational congruence $\simeq^{\epsilon}$, since all expressions of $\mathbb{E}$ that satisfy this equation (such as $recX(a.\tau.X)$ for instance) are related by $\simeq^{\epsilon}$.

But, if we consider $a = \tau$, various inequivalent expressions satisfy the equation $X = \tau.X$ (such as $\tau.0$ and $\tau.b.0$, for $b \neq \tau$). So, this equation is said to *not* have a unique solution modulo $\simeq^{\epsilon}$, and therefore Milner resorts to 'guarded' equations only (and treats unguarded expressions in a preprocessing step analogously to Theorem 4). In principle, the situation is not much different for $\simeq^{\Delta}$, where the equation $X = \tau.X$ does neither possess a unique solution, since $\Delta(E)$ satisfies it, for arbitrary $E \in \mathbb{E}$ (in other words and as in [10], the axiom (*rec*3) is only sound if restricted to guarded expressions). However, we cannot erase all divergence in a preprocessing step, simply because the relations considered are divergence sensitive, and thus divergence must somehow be kept during the entire completeness proof. To solve this problem we use Δ as a placeholder. It 'swallows' divergence whenever it arises during the transformations, and hence to ensure 'guardedness' even in the presence of divergence. Concretely, we use equation systems that treat Δ as a 'first class' citizen: For each variable X occuring in an equation, we provide a 'divergent copy'X^{Δ} together with the equation $X^{\Delta} = \Delta(X)$. With this twist, it is still a matter of precise bookkeeping to establish the proof.

Let $V \subseteq \mathbb{V}$ be a set of variables and let $\boldsymbol{X} = X_1, \dots, X_n$ be an ordered sequence of variables, where $X_i \notin V$. An *equation system over the free variables V and the formal variables* $\boldsymbol{X}$ is a set of equations $\mathcal{E} = \{X_i = E_i \mid 1 \leq i \leq n\}$ such that $E_i \in \mathbb{E}$ and $\mathbb{V}(E_i) \subseteq \{X_1, \dots, X_n\} \cup V$ for $1 \leq i \leq n$. Let $\boldsymbol{F} = F_1, \dots, F_n$ be an ordered sequence of expressions. Then $\boldsymbol{F}$ $*$-provably satisfies the equation system $\mathcal{E}$ if $F_i =^* E_i\{\boldsymbol{F}/\boldsymbol{X}\}$ for all $1 \leq i \leq n$. An expression F $*$-provably satisfies $\mathcal{E}$ if there exists a sequence of expressions $F_1, \dots, F_n$, which $*$-provably satisfies $\mathcal{E}$ and such that $F \equiv F_1$. We say that $\mathcal{E}$ is *guarded* if there exists a linear order $\prec$ on the variables $\{X_1, \dots, X_n\}$ such that whenever the variable X_j is unguarded in the expression E_i then $X_j \prec X_i$.

For the next definition we take for each formal variable X_i $(1 \leq i \leq n)$ a corresponding formal variable X_i^{Δ} such that $X_i^{\Delta} \notin \{X_1, \dots, X_n\} \cup V$. The symbols $\alpha, \beta, \gamma, \dots$ denote either Δ or $\_$. If e.g. $\alpha = \_$ then $X_i^{\alpha} \equiv X_i$ and $\alpha(E) \equiv E$. A *standard equation system (SES)* $\mathcal{E}$ over the free variables V and the formal variables $X_1, X_1^{\Delta}, \dots, X_n, X_n^{\Delta}$ is an equation system of the form

$$\mathcal{E} = \{X_i = E_i \mid 1 \leq i \leq n\} \cup \{X_i^{\Delta} = \Delta(X_i) \mid 1 \leq i \leq n\}$$

where E_i is a sum of expressions $a.X_j$ $(a \in \mathbb{A}, 1 \leq j \leq n)$, $\tau.X_j^{\Delta}$ $(1 \leq j \leq n)$, and variables $Y \in V$. We also say briefly that $\mathcal{E}$ is an SES over the free variables V and the formal variables $\boldsymbol{X} = X_1, \dots, X_n$. If the sequence $F_1, \Delta(F_1), \dots, F_n, \Delta(F_n)$ $*$-provably satisfies the SES $\mathcal{E}$ then we say briefly that $\boldsymbol{F} = F_1, \dots, F_n$ $*$-provably satisfies $\mathcal{E}$. Furthermore $E_i\{\boldsymbol{F}/\boldsymbol{X}\}$ denotes the expression that results from substituting in E_i the variable X_i^{α} by $\alpha(F_i)$, where $1 \leq i \leq n$ and $\alpha \in \{\_, \Delta\}$. We write $X_i^{\alpha} \xrightarrow{a}_{\mathcal{E}} X_j^{\beta}$ if E_i contains the summand $a.X_j^{\beta}$. Note that $X_i \xrightarrow{a}_{\mathcal{E}} X_j^{\beta}$ if and only if $X_i^{\Delta} \xrightarrow{a}_{\mathcal{E}} X_j^{\beta}$. The notions $X_i^{\alpha} \Longrightarrow_{\mathcal{E}} X_j^{\beta}$, $X_i^{\alpha} \stackrel{a}{\Longrightarrow}_{\mathcal{E}} X_j^{\beta}$, $X_i \not\rightarrow_{\mathcal{E}}, \dots$ are derived from the relations $\xrightarrow{a}_{\mathcal{E}}$ analogously to the corresponding notions in Section 3. If the SES $\mathcal{E}$ is clear from the context then we will omit the subscript $\mathcal{E}$ in the following. Note that $\mathcal{E}$ is guarded if and only if the relation $\xrightarrow{\tau}_{\mathcal{E}}$ is acyclic.

Finally, the SES $\mathcal{E}$ is *saturated* if for all $1 \le i, j \le n$ and α, β, if $X_i \stackrel{a}{\Longrightarrow} X_j^\alpha$ then also $X_i \stackrel{a}{\longrightarrow} X_j^\alpha$ (since we use this notion only for systems without free variables, we do not need Milner's saturation condition for free variables). The introduction of the new variables X_i^Δ and the special form of an SES is crucial in order to carry over Milner's saturation property in the presence of the Δ-operator:

Theorem 5. *Every guarded expression E $*$-provably satisfies a guarded and saturated SES over the free variables $\mathbb{V}(E)$.*

Using axiom (*rec*3), the following theorem can be shown analogously to [10].

Theorem 6. *Let $E, F \in \mathbb{E}$ and let $\mathcal{E}$ be a guarded equation system (not necessarily an SES) such that both E and F $*$-provably satisfy $\mathcal{E}$. Then $E =^* F$.*

6 Joining Two Equation Systems

In this section we restrict to expressions from $\mathbb{P}$. Our main technical result is

Theorem 7. *Let $P, Q \in \mathbb{P}$ such that $P \simeq^* Q$. Furthermore P (resp. Q) $*$-provably satisfies the guarded and saturated SES $\mathcal{E}_1 = \{X_i = E_i \mid 1 \le i \le m\}$ (resp. $\mathcal{E}_2 = \{Y_j = F_j \mid 1 \le j \le n\}$). Then there exists a guarded equation system $\mathcal{E}$ (not necessarily an SES) such that both P and Q $*$-provably satisfy $\mathcal{E}$.*

Let us postpone the proof of Theorem 7 for a moment and first see how completeness for $\mathbb{P}$ can be deduced:

Theorem 8 (completeness for $\mathbb{P}$). *If $P, Q \in \mathbb{P}$ and $P \simeq^* Q$ then $P =^* Q$.*

Proof. By Theorem 4 there exist guarded expressions P', Q' with $P' =^\Delta P$ and $Q' =^\Delta Q$. In particular, also $P', Q' \in \mathbb{P}$ and $P' \simeq^* Q'$ (due to soundness). By Theorem 5, P' (resp. Q') $*$-provably satisfies a guarded and saturated SES $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) without free variables. By Theorem 7 there is some guarded equation system $\mathcal{E}$ which is $*$-provably satisfied by P' and Q'. Theorem 6 gives $P' =^* Q'$, and hence $P =^* Q$, concluding the proof. $\square$

In order to prove Theorem 7, we need the following two lemmas.

Lemma 1. *Let $\mathcal{E}$ be a guarded SES over the formal variables $X_1, \ldots, X_n$, and let X_i be such that there do not exist k, α, and $a \in \mathbb{A} \setminus \{\tau\}$ with $X_i \stackrel{a}{\Longrightarrow}_{\mathcal{E}} X_k^\alpha$. Then there exist j, β with $X_i \Longrightarrow_{\mathcal{E}} X_j^\beta \not\rightarrow$.*

Proof. Induction along $\Longrightarrow_{\mathcal{E}}$, which is a partial order for a guarded SES. $\square$

For the further consideration it is useful to define a macro $\mathcal{M}^*(P)$ for $P \in \mathbb{P}$ by

$$\mathcal{M}^*(P) = \begin{cases} P \Uparrow & \text{if } * = \Delta, \\ P \stackrel{\tau}{\longrightarrow} & \text{if } * = S, \\ P \longrightarrow & \text{if } * = 0, \\ P \Lleftarrow\!\!\!\Uparrow & \text{if } * = \lambda. \end{cases}$$

Lemma 2. *If $\Delta(P) \sim^* \Delta(Q)$ then one of the following three cases holds:*

1. *$\mathcal{M}^*(P)$ and $P \sim^* \Delta(Q)$*
2. *$\mathcal{M}^*(Q)$ and $\Delta(P) \sim^* Q$*
3. *Neither $\mathcal{M}^*(P)$ nor $\mathcal{M}^*(Q)$, and $P \sim^* Q$*

Now we are able to prove Theorem 7.

Proof (Theorem 7). Assume that $\mathcal{E}_1$ is $*$-provably satisfied by the expressions $P_1, \dots, P_m \in \mathbb{P}$, where $P \equiv P_1$, and that $\mathcal{E}_2$ is $*$-provably satisfied by the expressions $Q_1, \dots, Q_n \in \mathbb{P}$, where $Q \equiv Q_1$. Thus $P_i =^* E_i\{\boldsymbol{P}/\boldsymbol{X}\}$ and $Q_j =^* F_j\{\boldsymbol{Q}/\boldsymbol{Y}\}$, and hence also $P_i \simeq^* E_i\{\boldsymbol{P}/\boldsymbol{X}\}$ and $Q_j \simeq^* F_j\{\boldsymbol{Q}/\boldsymbol{Y}\}$. Since $P, Q \in \mathbb{P}$, both $\mathcal{E}_1$ and $\mathcal{E}_2$ do not have free variables. The proof of the following two claims is tedious but straight-forward by using saturation of $\mathcal{E}_1$ and $\mathcal{E}_2$.

Claim 1 *If $\alpha(P_i) \sim^* \beta(Q_j)$ then the following implications hold:*

1. *If $X_i \xrightarrow{a} X_k^\gamma$ then either ($a = \tau$ and $\gamma(P_k) \sim^* \beta(Q_j)$) or there exist ℓ, δ such that $Y_j \xrightarrow{a} Y_\ell^\delta$ and $\gamma(P_k) \sim^* \delta(Q_\ell)$.*
2. *If $Y_j \xrightarrow{a} Y_\ell^\delta$ then either ($a = \tau$ and $\alpha(P_i) \sim^* \delta(Q_\ell)$) or there exist k, γ such that $X_i^\alpha \xrightarrow{a} X_k^\gamma$ and $\gamma(P_k) \sim^* \delta(Q_\ell)$.*
3. *Let $* = \Delta$. If $\alpha = \Delta$ then either $\beta = \Delta$ or $Y_j \xrightarrow{\tau} Y_\ell^\Delta$ for some ℓ.*
4. *Let $* = \Delta$. If $\beta = \Delta$ then either $\alpha = \Delta$ or $X_i \xrightarrow{\tau} X_k^\Delta$ for some k.*
5. *Let $* = \lambda$. If $\alpha = \Delta$ or ($\alpha = \_$ and $X_i \not\rightarrow$) then either $\beta = \Delta$, or ($\beta = \_$ and $Y_j \not\rightarrow$), or $Y_j \xrightarrow{\tau} Y_\ell^\Delta$ for some ℓ, or $Y_j \xrightarrow{\tau} Y_\ell \not\rightarrow$ for some ℓ.*
6. *Let $* = \lambda$. If $\beta = \Delta$ or ($\beta = \_$ and $Y_j \not\rightarrow$) then either $\alpha = \Delta$, or ($\alpha = \_$ and $X_i \not\rightarrow$), or $X_i \xrightarrow{\tau} X_k^\Delta$ for some k, or $X_i \xrightarrow{\tau} X_k \not\rightarrow$ for some k.*

Claim 2 *If $P_i \simeq^* Q_j$ then the following implications hold:*

1. *If $X_i \xrightarrow{a} X_k^\alpha$ then there exist ℓ, β such that $Y_j \xrightarrow{a} Y_\ell^\beta$ and $\alpha(P_k) \sim^* \beta(Q_\ell)$.*
2. *If $Y_j \xrightarrow{a} Y_\ell^\beta$ then there exist k, α such that $X_i \xrightarrow{a} X_k^\alpha$ and $\alpha(P_k) \sim^* \beta(Q_\ell)$.*

Now take for all $1 \le i \le m$, $1 \le j \le n$, and α, β with $\alpha(P_i) \sim^* \beta(Q_j)$ a variable $Z_{i,j}^{\alpha,\beta}$, and let $\boldsymbol{Z} = Z_{1,1}^{-,-}, \dots$ be a sequence consisting of these variables (since $P_1 \simeq^* Q_1$, $Z_{1,1}^{-,-}$ is defined). Moreover, if $\alpha(P_i) \sim^* \beta(Q_j)$ and either $\alpha = \_$ or $\beta = \_$ then we define $G_{i,j}^{\alpha,\beta}$ as the sum, which contains the summand

$$
\begin{array}{ll}
a.Z_{k,\ell}^{\gamma,\delta} & \text{if } X_i \xrightarrow{a} X_k^\gamma,\ Y_j \xrightarrow{a} Y_\ell^\delta, \text{ and } \gamma(P_k) \sim^* \delta(Q_\ell), \\
\tau.Z_{k,j}^{\gamma,\beta} & \text{if } X_i \xrightarrow{\tau} X_k^\gamma \text{ but } \neg\exists \ell, \delta : Y_j \xrightarrow{\tau} Y_\ell^\delta \ \wedge\ \gamma(P_k) \sim^* \delta(Q_\ell) \\
 & \text{(this implies by Claim 1(1) that } \gamma(P_k) \sim^* \beta(Q_j)), \\
\tau.Z_{i,\ell}^{\alpha,\delta} & \text{if } Y_j \xrightarrow{\tau} Y_\ell^\delta \text{ but } \neg\exists k, \gamma : X_i \xrightarrow{\tau} X_k^\gamma \ \wedge\ \gamma(P_k) \sim^* \delta(Q_\ell) \\
 & \text{(this implies by Claim 1(2) that } \alpha(P_i) \sim^* \delta(Q_\ell)).
\end{array}
$$

Furthermore $G_{i,j}^{\alpha,\beta}$ does not contain any other summands. Now the equation system $\mathcal{E}$ over the formal variables $\boldsymbol{Z}$ contains for each variable $Z_{i,j}^{\alpha,\beta}$ in $\boldsymbol{Z}$ the corresponding equation below, where for equation (E1) by Lemma 2 one of the three cases listed in (E1) holds (if the first and the second case hold, then we choose arbitrarily one of the two corresponding equations for (E1)).

$$\text{(E1)}\quad Z_{i,j}^{\Delta,\Delta} = \begin{cases} Z_{i,j}^{-,\Delta} & \text{if } \mathcal{M}^*(P_i) \text{ and } P_i \sim^* \Delta(Q_j) \\ Z_{i,j}^{\Delta,-} & \text{if } \mathcal{M}^*(Q_j) \text{ and } \Delta(P_i) \sim^* Q_j \\ \Delta(Z_{i,j}^{-,-}) & \text{if neither } \mathcal{M}^*(P_i) \text{ nor } \mathcal{M}^*(Q_j)\text{, and } P_i \sim^* Q_j \end{cases}$$

$$\text{(E2)}\quad Z_{i,j}^{\alpha,\beta} = \tau.G_{i,j}^{\alpha,\beta} \quad \text{if } \alpha = \Delta \neq \beta \text{ or } \alpha \neq \Delta = \beta$$

$$\text{(E3)}\quad Z_{i,j}^{-,-} = G_{i,j}^{-,-}$$

In general, $\mathcal{E}$ is not an SES, but from the guardedness of $\mathcal{E}_1$ and $\mathcal{E}_2$ it follows easily that also $\mathcal{E}$ is guarded. We will show that P $*$-provably satisfies $\mathcal{E}$, that also Q $*$-provably satisfies $\mathcal{E}$ can be shown analogously. For this we define for each variable $Z_{i,j}^{\alpha,\beta}$ in $\boldsymbol{Z}$ the corresponding expression $R_{i,j}^{\alpha,\beta}$ by

$$R_{i,j}^{\Delta,\Delta} \equiv R_{i,j}^{\Delta,-} \equiv \Delta(P_i), \qquad R_{i,j}^{-,\Delta} \equiv \tau.P_i, \text{ and}$$

$$R_{i,j}^{-,-} \equiv \begin{cases} P_i & \text{if } \forall \ell,\delta,a \left\{ Y_j \xrightarrow{a} Y_\ell^\delta \;\Rightarrow\; \exists k,\gamma \left\{ \begin{array}{l} X_i \xrightarrow{a} X_k^\gamma \wedge \\ \gamma(P_k) \sim^* \delta(Q_\ell) \end{array} \right\} \right\} \\ \tau.P_i & \text{if } \exists \ell,\delta \left\{ Y_j \xrightarrow{\tau} Y_\ell^\delta \;\wedge\; \neg\exists k,\gamma \left\{ \begin{array}{l} X_i \xrightarrow{\tau} X_k^\gamma \wedge \\ \gamma(P_k) \sim^* \delta(Q_\ell) \end{array} \right\} \right\} \end{cases}$$

Let $\boldsymbol{R} = R_{1,1}^{-,-}, \ldots$ be the sequence corresponding to the sequence $\boldsymbol{Z}$. First note that $R_{1,1}^{-,-} \equiv P_1 \equiv P$ by $P_1 \simeq^* Q_1$ and Claim 2(2). It remains to check that all equations are $*$-provably satisfied when every variable $Z_{i,j}^{\alpha,\beta}$ is replaced by $R_{i,j}^{\alpha,\beta}$. We start with equation (E1) defining $Z_{i,j}^{\Delta,\Delta}$. The case that $Z_{i,j}^{\Delta,\Delta}$ is defined by $Z_{i,j}^{\Delta,\Delta} = Z_{i,j}^{\Delta,-}$ is trivial, since $R_{i,j}^{\Delta,\Delta} \equiv R_{i,j}^{\Delta,-} \equiv \Delta(P_i)$. Thus, the following two cases 1 and 2 remain.

Case 1. Equation $Z_{i,j}^{\Delta,\Delta} = Z_{i,j}^{-,\Delta}$ belongs to $\mathcal{E}$: thus $\mathcal{M}^*(P_i)$ and $P_i \sim^* \Delta(Q_j)$. Since $R_{i,j}^{-,\Delta} \equiv \tau.P_i$ and $R_{i,j}^{\Delta,\Delta} \equiv \Delta(P_i)$ we have to prove that $\tau.P_i =^* \Delta(P_i)$. We distinguish on the value of $*$.

Case 1.1. $* = \Delta$: then $P_i \sim^\Delta \Delta(Q_j)$ and hence $X_i \xrightarrow{\tau} X_k^\Delta$ for some k by Claim 1(4). Thus there exists an expression R with (we use the derived law $(\tau\Delta')$ from Section 4) $\tau.P_i =^\Delta \tau.E_i\{\boldsymbol{P}/\boldsymbol{X}\} =^\Delta \tau.(R+\tau.\Delta(P_k)) =^\Delta \tau.(R+\Delta(P_k)) =^\Delta \Delta(R+\Delta(P_k)) =^\Delta \cdots =^\Delta \Delta(P_i)$.

Case 1.2. $* = S$: then $P_i \sim^S \Delta(Q_j)$ and $\mathcal{M}^S(P_i)$, i.e., $P_i \xrightarrow{\tau}$. Since $P_i \simeq^S E_i\{\boldsymbol{P}/\boldsymbol{X}\}$, also $E_i\{\boldsymbol{P}/\boldsymbol{X}\} \xrightarrow{\tau}$, i.e, $X_i \xrightarrow{\tau}$, and there exist expressions R, P_k with $\tau.P_i =^S \tau.E_i\{\boldsymbol{P}/\boldsymbol{X}\} =^\Delta \tau.(R+\tau.P_k) =^S \Delta(R+\tau.P_k) =^S \cdots =^S \Delta(P_i)$.

Case 1.3. $* = 0$: analogously to Case 1.2 with axiom (0) used instead of (S).

Case 1.4. $* = \lambda$: then $P_i \sim^\lambda \Delta(Q_j)$, and Claim 1(6) implies either $X_i \not\xrightarrow{}$, or $X_i \xrightarrow{\tau} X_k^\Delta$ for some k, or $X_i \xrightarrow{\tau} X_k \not\xrightarrow{}$ for some k.

Case 1.4.1. $X_i \not\xrightarrow{}$, i.e., $E_i \equiv 0$: [1] we obtain $\tau.P_i =^\lambda \tau.E_i\{\boldsymbol{P}/\boldsymbol{X}\} \equiv \tau.0 =^\lambda \Delta(0) =^\lambda \cdots =^\lambda \Delta(P_i)$.

Case 1.4.2. $X_i \xrightarrow{\tau} X_k^\Delta$: we can conclude as in Case 1.1.

Case 1.4.3. $X_i \xrightarrow{\tau} X_k \not\xrightarrow{}$: thus there exists R with $\tau.P_i =^\lambda \tau.E_i\{\boldsymbol{P}/\boldsymbol{X}\} =^\Delta \tau.(R + \tau.0) =^\lambda \tau.(R + \Delta(0)) =^\Delta \Delta(R + \Delta(0)) =^\lambda \cdots =^\lambda \Delta(P_i)$.

Case 2. Equation $Z_{i,j}^{\Delta,\Delta} = \Delta(Z_{i,j}^{-,-})$ belongs to $\mathcal{E}$: thus $P_i \sim^* Q_j$ and neither $\mathcal{M}^*(P_i)$ nor $\mathcal{M}^*(Q_j)$ holds. We have either $R_{i,j}^{-,-} \equiv P_i$ or $R_{i,j}^{-,-} \equiv \tau.P_i$. The case that $R_{i,j}^{-,-} \equiv P_i$ is trivial, thus let us assume that $R_{i,j}^{-,-} \equiv \tau.P_i$. Then there exist ℓ, δ such that $Y_j \xrightarrow{\tau} Y_\ell^\delta$ but there do not exist k, γ with $X_i \xrightarrow{\tau} X_k^\gamma$ and $\gamma(P_k) \sim^* \delta(Q_\ell)$. Since $\Delta(P_i) \sim^* \Delta(Q_j)$ (recall that variable $Z_{i,j}^{\Delta,\Delta}$ is defined), it follows $\Delta(P_i) \sim^* \delta(Q_\ell)$ by Claim 1(2). Using Claim 1 we can deduce for each value of $*$ a contradiction to $\neg\mathcal{M}^*(Q_j)$.

It remains to check the equations (E2) and (E3). Fix α, β such that $\alpha(P_i) \sim^* \beta(Q_j)$ and either $\alpha = \_$ or $\beta = \_$. We will distinguish two main cases 3 and 4:

Case 3. $\forall \ell, \delta, a\ (Y_j \xrightarrow{a} Y_\ell^\delta \Rightarrow \exists k, \gamma : X_i \xrightarrow{a} X_k^\gamma \wedge \gamma(P_k) \sim^* \delta(Q_\ell))$ (†)

With axiom (τ1) and (S1)-(S3) we obtain $G_{i,j}^{\alpha,\beta}\{\boldsymbol{R}/\boldsymbol{Z}\} =^\Delta E_i\{\boldsymbol{P}/\boldsymbol{X}\} =^* P_i$ (this step is analogous to [10]). In case $\alpha = \_ = \beta$ (resp. $\alpha = \_$, $\beta = \Delta$), it is straight-forward to show that equation (E3) (resp. (E2)) is satisfied. So assume that $\alpha = \Delta$, $\beta = \_$. Thus $\Delta(P_i) \sim^* Q_j$. By inspecting equation (E2) and using the fact that $R_{i,j}^{\Delta,-} \equiv \Delta(P_i)$ and $G_{i,j}^{\alpha,\beta}\{\boldsymbol{R}/\boldsymbol{Z}\} =^* P_i$, we see that it remains to show $\Delta(P_i) =^* \tau.P_i$. We distinguish on the value of $*$.

Case 3.1. $* = \Delta$: thus $\Delta(P_i) \sim^\Delta Q_j$ and $Y_j \xrightarrow{\tau} Y_\ell^\Delta$ for some ℓ by Claim 1(3). Hence by (†) there exist k, γ with $X_i \xrightarrow{\tau} X_k^\gamma$ and $\gamma(P_k) \sim^\Delta \Delta(Q_\ell)$. By Claim 1(4) either $\gamma = \Delta$ or $X_k \xrightarrow{\tau} X_p^\Delta$ for some p. Saturation of $\mathcal{E}_1$ implies in both cases $X_i \xrightarrow{\tau} X_p^\Delta$ for some p, which allows to conclude as in Case 1.1.

Case 3.2. $* = S$: we have $\Delta(P_i) \sim^S Q_j$. If $Q_j \not\xrightarrow{\tau}$ then $P_i \xrightarrow{\tau}$, and we can refer to Case 1.2. On the other hand, if $Q_j \xrightarrow{\tau}$ then $F_j\{\boldsymbol{Q}/\boldsymbol{Y}\} \xrightarrow{\tau}$, i.e, $Y_j \xrightarrow{\tau}$. Thus $X_i \xrightarrow{\tau}$ by (†), which allows again to refer to Case 1.2.

Case 3.3. $* = 0$: analogous to Case 3.2.

Case 3.4. $* = \lambda$: since $\Delta(P_i) \sim^\lambda Q_j$, Claim 1(5) implies either $Y_j \not\xrightarrow{}$, or $Y_j \xrightarrow{\tau} Y_\ell^\Delta$, or $Y_j \xrightarrow{\tau} Y_\ell \not\xrightarrow{}$ for some ℓ.

Case 3.4.1. $Y_j \not\xrightarrow{}$: by Claim 1(1) there cannot exist $a \in \mathbb{A} \setminus \{\tau\}$ with $X_i \xRightarrow{a}$. Lemma 1 and the saturation of $\mathcal{E}_1$ imply either $X_i \not\xrightarrow{}$, or $X_i \xrightarrow{\tau} X_k^\Delta$ for some k, or $X_i \xrightarrow{\tau} X_k \not\xrightarrow{}$ for some k. We can proceed as in Case 1.4.

[1] Note that if we would deal with equation systems containing free variables, then we could only conclude here that E_i must be a sum of free variables. This is the reason why Theorem 7 requires that $P, Q \in \mathbb{P}$, i.e., that $\mathbb{V}(P) = \mathbb{V}(Q) = \emptyset$.

Case 3.4.2. $Y_j \xrightarrow{\tau} Y_\ell^\Delta$ for some ℓ: by (†) there exist k, γ with $X_i \xrightarrow{\tau} X_k^\gamma$ and $\gamma(P_k) \sim^\lambda \Delta(Q_\ell)$. By Claim 1(6) either $\gamma = \Delta$, or ($\gamma = \_$ and $X_k \not\rightarrow$), or $X_k \xrightarrow{\tau} X_p^\Delta$ for some p, or $X_k \xrightarrow{\tau} X_p \not\rightarrow$ for some p. By saturation we obtain either $X_i \xrightarrow{\tau} X_p^\Delta$ for some p (see Case 1.4.2), or $X_i \xrightarrow{\tau} X_p \not\rightarrow$ for some p (see Case 1.4.3).

Case 3.4.3. $Y_j \xrightarrow{\tau} Y_\ell \not\rightarrow$ for some ℓ: by (†) there exist k, γ with $X_i \xrightarrow{\tau} X_k^\gamma$ and $\gamma(P_k) \sim^\lambda Q_\ell$. Using Claim 1(6) we can conclude as in Case 3.4.2.

Case 4. $\exists \ell, \delta\ (Y_j \xrightarrow{\tau} Y_\ell^\delta \ \wedge\ \neg\exists k, \gamma : X_i \xrightarrow{\tau} X_k^\gamma \ \wedge\ \gamma(P_k) \sim^* \delta(Q_\ell))$

We get $G_{i,j}^{\alpha,\beta}\{\boldsymbol{R}/\boldsymbol{Z}\} =^\Delta E_i\{\boldsymbol{P}/\boldsymbol{X}\} + \tau.\alpha(P_i) =^* P_i + \tau.\alpha(P_i)$ (as in Case 3, this step is analogous to [10]).

Case 4.1. $\alpha = \beta = \_$: we have $R_{i,j}^{\_;\_} \equiv \tau.P_i =^\Delta P_i + \tau.P_i =^* G_{i,j}^{\_;\_}\{\boldsymbol{R}/\boldsymbol{Z}\}$, thus (E3) is satisfied.

Case 4.2. $\alpha = \_,\ \beta = \Delta$: we obtain $R_{i,j}^{\_;\Delta} \equiv \tau.P_i =^\Delta \tau.\tau.P_i =^\Delta \tau.(P_i + \tau.P_i) =^* \tau.G_{i,j}^{\_;\Delta}\{\boldsymbol{R}/\boldsymbol{Z}\}$, thus (E2) is satisfied.

Case 4.3. $\alpha = \Delta,\ \beta = \_$: with $(\tau\Delta')$ and $(\tau\Delta)$ from Section 4 we get $R_{i,j}^{\Delta;\_} \equiv \Delta(P_i) =^\Delta \tau.\Delta(P_i) =^\Delta \tau.(P_i + \tau.\Delta(P_i)) =^* \tau.G_{i,j}^{\Delta;\_}\{\boldsymbol{R}/\boldsymbol{Z}\}$, thus (E2) is again satisfied. This concludes the proof of Theorem 7 and hence of Theorem 8. □

7 Completeness for Open Expressions

In order to prove completeness for the whole set $\mathbb{E}$ we will argue in a purely syntactical way by investigating our axioms. The following observation is crucial:

Lemma 3. *Let $* \neq 0$ and $E, F \in \mathbb{E}$. If $a \in \mathbb{A} \setminus \{\tau\}$ does neither occur in E nor in F then $E\{a.0/X\} =^* F\{a.0/X\}$ implies $E =^* F$.*

Note that Lemma 3 is false for $* = 0$. We have $\tau.a.0 =^0 \Delta(a.0)$ but $\tau.X \neq^0 \Delta(X)$ (since $\tau.0 \not\simeq^0 \Delta(0)$). Hence, in the following theorem we have to exclude $* = 0$.

Theorem 9. *Let $* \neq 0$ and $E, F \in \mathbb{E}$. If $E \simeq^* F$ then $E =^* F$.*

Proof. Let $E \simeq^* F$. We prove by induction on $|\mathbb{V}(E) \cup \mathbb{V}(F)|$ that $E =^* F$. If $\mathbb{V}(E) \cup \mathbb{V}(F) = \emptyset$ then in fact $E, F \in \mathbb{P}$ and $E =^* F$ by Theorem 8. Thus let $X \in \mathbb{V}(E) \cup \mathbb{V}(F)$. Since $E \simeq^* F$, we have $E\{a.0/X\} \simeq^* F\{a.0/X\}$. Thus by induction $E\{a.0/X\} =^* F\{a.0/X\}$ and hence $E =^* F$ by Lemma 3. □

In order to obtain completeness for $\simeq^0$ on open expressions, we have to introduce the following additional axiom ($\mathbb{E}$), which can be shown to be sound for $\simeq^0$.

($\mathbb{E}$) If $E\{0/X\} = F\{0/X\}$ and $E\{a.0/X\} = F\{a.0/X\}$ where $a \in \mathbb{A} \setminus \{\tau\}$ does neither occur in E nor in F then $E = F$.

If we add this axiom to the standard axioms for $\simeq^0$ then we can prove completeness in the same way as in the proof of Theorem 9.

Theorem 10. *Let $E, F \in \mathbb{E}$. If $E \simeq^0 F$ then $E =^0 F$ can be derived by the standard axioms for $\simeq^0$ plus the axiom ($\mathbb{E}$).*

8 Conclusion

This paper has developed sound and complete axiomatisations for the divergence sensitive spectrum of weak bisimulation equivalences. We have not covered the weak bisimulation preorders $WB^{\downarrow}$ and $WB^{\downarrow\downarrow}$ considered in [5]. We claim however that adding the axiom $\Delta(E) \leq E + F$ to the axioms of WB^{λ} (respectively WB^{Δ}) is enough to obtain completeness of $WB^{\downarrow\downarrow}$ ($WB^{\downarrow}$). Note that $WB^{\downarrow}$ is axiomatised in [11], so only $WB^{\downarrow\downarrow}$ needs further work.

We are confident that our axiomatisation form the basis of a complete equational characterisation of the bisimulation fragment of the linear time – branching time spectrum with silent moves. On the technical side, we are currently investigating whether the somewhat unsatisfactory auxiliary axiom ($\mathbb{E}$) is indeed necessary for achieving completeness of open expressions for $\simeq^0$.

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